Charging a Capacitor via a Transient RLC Circuit

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1 Problem

Discuss the time evolution of various forms of energy a series RLC circuit that is energized at time $t = 0$ by a battery of voltage V. Include consideration of radiated energy, supposing that the circuit has the form of a circular loop of radius a.

This problem relates to the question of whether a capacitor can be charged without loss of energy. As confirmed in sec. 2.1, if the capacitor is charged to voltage V in a simple RC circuit, then the resistor dissipates energy equal to that eventually stored in the capacitor. Heinrich [1] noted that this energy loss could be avoided if the battery is replaced by a variable power supply whose voltage is raised "slowly" to the desired value V.[1] See also [2, 3]. If one capacitor is charged by another in a circuit with negligible resistance, there is again a loss of energy, to radiation in this case [4, 5, 6].

These analyses leave open the question of whether energy loss is inevitable whenever a capacitor is charged "quickly". Show that a capacitor can be charged with only modest energy loss in an underdamped series RLC circuit if the battery is disconnected after 1/2 cycle.

2 Solution

Energy flows from the battery into four forms: the I^2R heating of the resistor, the electrostatic energy $U_C = CV^2/2$ that remains stored in the capacitor once the transient current has died out, the energy $U_L(t) = LI^2/2$ that is temporarily stored in the inductor while the current is nonzero, and the energy radiated away while the current in the circuit is changing. We assume that radius a of the circuit is small compared to the wavelength of all significant frequency components of the radiation, so that the current I is independent of position around the circuit and the radiation is well approximated as that associated with the magnetic dipole moment

$$m(t) = \pi a^2 I(t), \tag{1}$$

namely

$$\frac{dU_{\rm rad}}{dt} = \frac{1}{6\pi c^4}\sqrt{\frac{\mu_0}{\epsilon_0}}\ddot{m}^2 = 2.4 \times 10^{-32} a^4 \ddot{I}^2. \tag{2}$$

The Kirchhoff equation for the series RLC circuit is

$$V = L\dot{I} + IR + \frac{Q}{C}\,, \tag{3}$$

[1]The stored energy is $Q^2/2C \propto [\int I\,dt]^2$, while the energy dissipated is $\int I^2R\,dt$. So if the current I is smaller and lasts for a longer time, the stored energy can be the same but the energy dissipated will be less. To obtain a lower current in the circuit, the voltage applied during the characteristic time interval for energy dissipation must be smaller; hence the prescription to raise the voltage slowly.

whose time derivative is

$$0 = L\ddot{I} + \dot{I}R + \frac{I}{C}\,. \tag{4}$$

We seek solutions of the form $e^{-\alpha t}$, for which eq. (4) leads to the quadratic equation

$$L\alpha^2 - R\alpha + \frac{1}{C} = 0, \tag{5}$$

whose solutions are

$$\alpha_{1,2} = \frac{R}{2L} \mp \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{R}{2L} \mp i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,. \tag{6}$$

The current in the circuit is zero at time $t = 0$ when the battery is connected to the circuit (and it cannot jump instantaneously to a nonzero value because of the inductor). Hence, the total current in the circuit can be written

$$I(t) = I_0(e^{-\alpha_1 t} - e^{-\alpha_1 t}) = 2I_0 e^{-Rt/2L} \sinh\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}t = 2iI_0 e^{-Rt/2L} \sin\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}t. \tag{7}$$

Just after the battery is connected, the voltage drops across the resistor and capacitor are still zero, so the initial voltage drop across the inductor is related by

$$V = L\dot{I}(0) = LI_0(\alpha_2 - \alpha_1) = I_0\sqrt{R^2 - \frac{4L}{C}} = iI_0\sqrt{\frac{4L}{C} - R^2}\,. \tag{8}$$

We now consider the cases that R is larger or smaller than $2\sqrt{L/C}$.

2.1 Overdamped Circuit: $R > 2\sqrt{L/C}$

In this case the current is given by

$$I(t) = \frac{V}{\sqrt{R^2 - \frac{4L}{C}}}(e^{-\alpha_1 t} - e^{-\alpha_2 t}) = \frac{V}{\sqrt{\frac{R^2}{4} - \frac{L}{C}}} e^{-Rt/2L} \sinh\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}t. \tag{9}$$

The energy temporarily stored in the inductor at time t is

$$U_L(t) = \frac{LI^2}{2} = \frac{V^2 L}{\frac{R^2}{2} - \frac{2L}{C}} e^{-Rt/L} \sinh^2\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}t. \tag{10}$$

For large resistance R the inductive energy reaches a maximum of $U_{L,\max} \approx U_C R/\sqrt{L/C} \gg U_C$ at time $t \approx (L/R)\ln(R^2C/L)$.

The power dissipated in the resistor is

$$\frac{dU_{\text{Joule}}}{dt} = I^2 R = \frac{V^2 R}{R^2 - \frac{4L}{C}}(e^{-2\alpha_1 t} - 2e^{-(\alpha_1+\alpha_2)t} + e^{-2\alpha_2 t}), \tag{11}$$

and the total energy dissipated after a long time is

$$U_{\rm Joule} = \frac{V^2 R}{R^2 - \frac{4L}{C}} \left(\frac{1}{2\alpha_1} - \frac{2}{\alpha_1 + \alpha_2} + \frac{1}{2\alpha_2} \right) = \frac{V^2 R}{R^2 - \frac{4L}{C}} \left(\frac{RC}{2} - \frac{2L}{R} \right) = \frac{CV^2}{2} = U_C, \qquad (12)$$

where $U_C = CV^2/2$ is the energy stored in the capacitor at large time t.

The radiated power is obtained from eqs. (2) and (9),

$$\frac{dU_{\rm rad}}{dt} = 2.4 \times 10^{-32} a^4 \frac{V^2}{R^2 - \frac{4L}{C}} (\alpha_1^2 e^{-2\alpha_1 t} - 2\alpha_1\alpha_2 e^{-(\alpha_1+\alpha_2)t} + \alpha_2^2 e^{-2\alpha_2 t}), \qquad (13)$$

and the total radiated power after a long time is

$$U_{\rm rad} = 2.4 \times 10^{-32} a^4 \frac{V^2}{R^2 - \frac{4L}{C}} \left(\frac{\alpha_1}{2} - \frac{2\alpha_1\alpha_2}{\alpha_1 + \alpha_2} + \frac{\alpha_2}{2} \right) = 2.4 \times 10^{-32} a^4 \frac{U_C}{RLC}\,. \qquad (14)$$

In principle the radiated energy can become large if the inductance is very small such that the second derivative $\ddot{I}$ becomes very large. However, the inductance of a loop of radius a made of wire of radius b is $L \approx \mu_0 a \ln(a/b)$, so the radiated power is bounded by

$$U_{\rm rad} \lesssim 3 \times 10^{-38} a^5 \ln\frac{a}{b} \frac{1}{RC} U_C \qquad (15)$$

(in SI units). In any practical, transient RLC circuit the radiated energy is negligible.

In sum, when a capacitor is charged via an overdamped RLC circuit, as much energy is lost to Joule heating as ends up stored in the capacitor.

2.2 Underdamped Circuit: $R < 2\sqrt{L/C}$

In this case the current is given by

$$I(t) = \frac{V}{i\sqrt{\frac{4L}{C} - R^2}} (e^{-\alpha_1 t} - e^{-\alpha_2 t}) = \frac{V}{\omega L} e^{-Rt/2L} \sin\omega t, \qquad (16)$$

where

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}. \qquad (17)$$

The energy temporarily stored in the inductor at time t is

$$U_L(t) = \frac{LI^2}{2} = \frac{V^2}{2\omega^2 L} e^{-Rt/L} \sin^2\omega t. \qquad (18)$$

For small resistance R the inductive energy reaches a maximum of $U_{L,\rm max} \approx U_C$ at time $t \approx \pi/2\omega \approx \pi\sqrt{LC}/2$.

The charge $Q(t)$ on the capacitor at time t is

$$\begin{aligned} Q(t) &= \int_0^t I(t)\,dt = \frac{V}{\omega^2 L} \int_0^{\omega t} e^{-Rx/2\omega L} \sin x\,dx \\ &= \frac{V}{\omega^2 L} \frac{1}{1 + R^2/4\omega^2 L^2} \left[1 - e^{-Rt/2L} \left(\frac{R}{2\omega L} \sin\omega t + \cos\omega t \right) \right] \\ &= VC \left[1 - e^{-Rt/2L} \left(\frac{R}{2\omega L} \sin\omega t + \cos\omega t \right) \right]. \qquad (19) \end{aligned}$$

The energy $U_C(t)$ stored in the capacitor at time t is

$$U_C(t) = \frac{Q^2(t)}{2C} = U_C \left[1 - e^{-Rt/2L}\left(\frac{R}{2\omega L}\sin\omega t + \cos\omega t\right)\right]^2. \tag{20}$$

The power dissipated in the resistor is

$$\frac{dU_{\rm Joule}}{dt} = I^2 R = \frac{V^2 R}{\omega^2 L^2} e^{-Rt/L} \sin^2 \omega t, \tag{21}$$

and the energy $U_{\rm Joule}(t)$ dissipated in the resistor up to time t is

$$\begin{aligned} U_{\rm Joule}(t) &= \frac{V^2 R}{\omega L^2} \int_0^{\omega t} e^{-Rx/\omega L} \sin^2 x \, dx \\ &= U_C \left[1 - e^{-Rt/L}\left(1 + \frac{R^2 \sin^2 \omega t}{2\omega^2 L^2} + \frac{R}{2\omega L}\sin 2\omega t\right)\right]. \end{aligned} \tag{22}$$

For large t the energy dissipated equals the energy stored. However, the battery could be disconnected from the circuit whenever the current is zero, *i.e.*, at $t = n\pi/\omega$. In particular, if the battery were disconnected at time $t = \pi/\omega$, we would have

$$\frac{U_{\rm Joule}(\pi/\omega)}{U_C(\pi/\omega)} = \frac{1 - e^{-\pi R/\omega L}}{1 + e^{-\pi R/2\omega L}} \approx \frac{\pi R}{2\sqrt{L/C}}, \tag{23}$$

where the approximation holds for small resistance R. That is, the capacitor can be charged with only small loss of energy to Joule heating by use of a large L, small R, and connecting the battery for only 1/2 of a (damped) cycle. As a bonus, the resulting voltage on the capacitor is nearly twice that of the battery.

When $R \ll \sqrt{L/C}$ the second time derivative of the current is

$$\ddot{I}(t) \approx \frac{V\omega}{L} e^{-Rt/2L} \sin \omega t. \tag{24}$$

The radiated power is obtained from eqs. (2) and (24),

$$\frac{dU_{\rm rad}}{dt} \approx 2.4 \times 10^{-32} a^4 \frac{V^2 \omega^2}{L^2} e^{-Rt/L} \sin^2 \omega t, \tag{25}$$

and the total radiated power up to time t is

$$U_{\rm rad}(t) \approx 2.4 \times 10^{-32} a^4 \frac{U_C}{RLC}(1 - e^{-Rt/L}) \tag{26}$$

Then,

$$U_{\rm rad}(\pi/\omega) \approx 2 \times 10^{-32} a^4 \frac{\pi U_C}{L\sqrt{LC}} \ll U_C \tag{27}$$

Again, the radiation in this transient RLC circuit is negligible.

In sum, while a capacitor that is charged for long times in an underdamped RLC circuit stores only as much energy as is lost to Joule heating, if the battery is disconnected after 1/2 cycle, the stored energy can be large compared to the energy lost to heat and radiation.

Acknowledgment

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