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The toroid antenna as a conditioner of electromagnetic fields into (low energy) gauge fields*

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Treatment of the radiated field from a toroid antenna as two A fields in resonance or a Φ field, and the toroid as a Φ field radiator, results in the prediction of (i) omnidirectional radiation patterns (with small indentations at the poles), and (ii) periodic resonances in the driving conditions. These predictions have been confirmed experimentally, giving validity to the fundamental nature of this topological and group theory understanding of the first order determinants of electromagnetic field dynamics. Resonant gauge (Φ) fields are produced by a toroid radiator as either propagating or standing waves. In the case of a torus with a single winding, an alternating current driver will produce a series of nonmeasurable A vector potential resonances, which overlap and combine into measurable phase factor or gauge field, Φ, waves. Such Φ waves, although generated on a toroidal-solenoidal structure of nonsimple topology, are yet spherical waves – either standing spherical waves, or propagating spherical waves. The topological constraints of electromagnetic fields mapped to a torus driven by single or double wiring are described. It is shown that such mapping results in group symmetries higher than $U(1)$, e.g., $SU(2)$, and that electromagnetic activity is affected by such mapping, being determined by the symmetry forms. Underpinning these symmetry forms are topological conservation laws. The toroid antenna exhibits a series of low and high impedances and permits a $U(1)$ to $SU(2)$ mapping of e.m. fields over a fiber bundle, as well as a mapping of rational and real numbers to complex numbers (in S^3 for the *nonresonant* condition) and quaternions (in S^4 for the *resonant* condition). When in resonance, the singly-wound and the doubly-wound (caduceous) torus emits radiation in $SU(2)/Z_2$ form. The fields emitted for the resonantly driven toroid are alternatively self-dual and anti-self-dual (i.e., instanton solutions to the Maxwell equations of S^4). In resonance, the singly wound and the doubly wound torus produce fields which are both *multiply connected*, and of $SU(2)/Z_2$ form (homeomorphic to S^3), as well as *simply connected*, and of $SU(2)$ from (homeomorphic to S^4). This study has implications far beyond the immediate subject. If the conventional theory of electromagnetism, i.e., 'Maxwell's theory', which is of $U(1)$ symmetry form, is but the simplest local theory of electromagnetism, then those pursuing a unified field theory may wish to consider as a candidate field for unification not only this simple local theory, but other forms of 'conditioned' electromagnetism. As is shown here, other such forms can be either force fields or gauge fields of higher group symmetry, e.g., $SU(2)$ and above.

Introduction

In topology, the torus represents a 2-to-1 mapping of a $U(1)$ field into an $SU(2)$ field. Maxwell's theory addresses local, $U(1)$ symmetry fields, not $SU(2)$ fields [3–7]. Therefore,

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it is instructive to consider the fields produced around a toroidally-wound solenoid, the torus being the Cartesian product of two circles, one determining a latitude, the other a longitude. We commence the discussion by considering a linear solenoid (Fig. 1), in which the lines of force (the H field) for adjacent wires annihilate each other except parallel to the surface of the coil on its inside and its outside. It is well known that this field consists of lines of force parallel to the axis of the coil running inside the coil and outside the coil.

If the coil, with N turns, is bent into a circle of radius, R, with the two ends together, the situation changes (Fig. 2). In this torus configuration, no lines of force emerge from the coil but exist continuously inside the coil.

It is useful now to consider the coil and the ring in this torus configuration separately. The current behaves differently in a tightly wound state. Loop currents produce current sheets forming around the ring. With close spacing of the solenoid wires, the current sheet becomes continuous. Whereas the toroidal coil has a sheet current density of

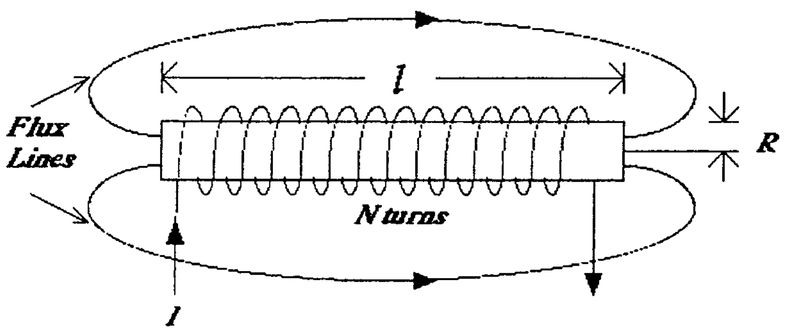

Fig. 1. Linear solenoid.

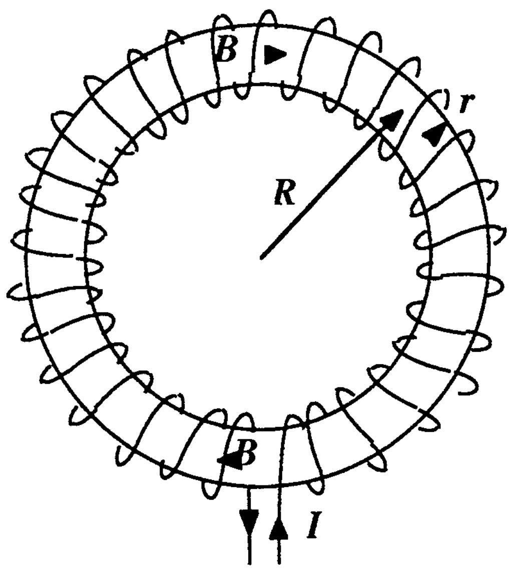

Fig. 2. Toroidal solenoid.

$$K = \frac{NI}{2\pi R} \text{ (A/m)}, \tag{1}$$

the ring has an equivalent current surface:

$$K' = 4K \text{ (A/m)}. \tag{2}$$

Next, the function of the toroidal solenoid depends on the physical composition of the ring. We consider two examples: (1) the ring has no ferromagnetic materials so that the magnetization, $M = 0$; and (2) the ring is composed of ferromagnetic materials, so $M \neq 0$.

In case (1), the magnetic flux density is:

$$B = \frac{\mu_0 NI}{2\pi R} = \mu_0 R \text{ (Wb/m}^2\text{)}, \tag{3}$$

where μ_0 is permeability.

The magnetic field for this case is:

$$H = \frac{B}{\mu_0} = \frac{NI}{2\pi R} = K \text{ (A/m)}, \tag{4}$$

That is, for this case the magnetic field is equal to the sheet current density of the coil winding. For this case also, both B and H are continuous and have the same direction.

The situation changes for case (2), i.e., when the ring is a ferromagnetic material, e.g., an iron ring. The total magnetization is then:

$$M = K' = 4K \text{ (A/m)}, \tag{5}$$

and the magnetic flux density is:

$$B = \mu_0(K + K') = 5\mu_0 K \text{ (Wb/m)}, \tag{6}$$

The magnetic field is:

$$H = \frac{B}{\mu_0} - M \cong K \text{ (A/m)}, \tag{7}$$

and H and B have the same direction.

Thus, the magnetic flux density will vary by a factor of 5 depending on whether the ring of the toroidal solenoid is nonferromagnetic, e.g., Styrofoam, or ferromagnetic, e.g., iron. In the following, we assume that the ring is Styrofoam.

Assuming then a nonferromagnetic ring and using the well known relation between the magnetic flux density and the vector potential, A:

$$B = \nabla \times A, \tag{8}$$

where the vector potential field is defined:

$$A(r) = \frac{1}{c} \iiint \frac{J(r)}{|r|} \mathrm{d}^3 r, \tag{9}$$

and J is the current density, we can plot the isopotential lines around and through the toroidal solenoid (Fig. 3). The E and the B fields within the ring of the solenoid are shown in Fig. 4.

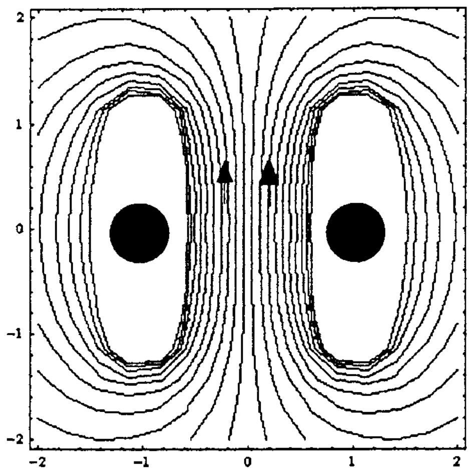

Fig. 3. Plot of a cut through A potential field lines surrounding a toroidal solenoid represented as two black dots which join out of the page and into the page.

Discussion

All the above is well known. Addressing, now, the absolute difference of two A potentials of opposite polarity generated on a torus, i.e., the A field generated on a torus which overlaps one half cycle with itself, some novel observations can be made.

Referring to Fig. 5, the toroidal solenoid is viewed now as a space for mapping two A potential alternating patterns ϕ_1 and ϕ_2 onto each other thereby generating a differential phase factor wave Φ. It is this phase factor wave Φ which is either formed around the

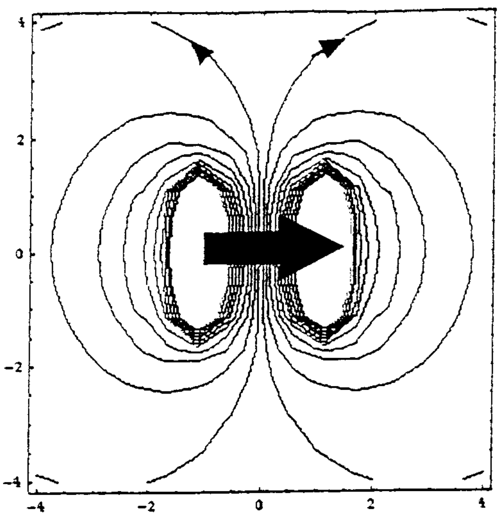

Fig. 4. A second plot of a cut through A potential field lines surrounding a toroidal solenoid. The direction of the B field in the torus ring is represented by the black arrow.

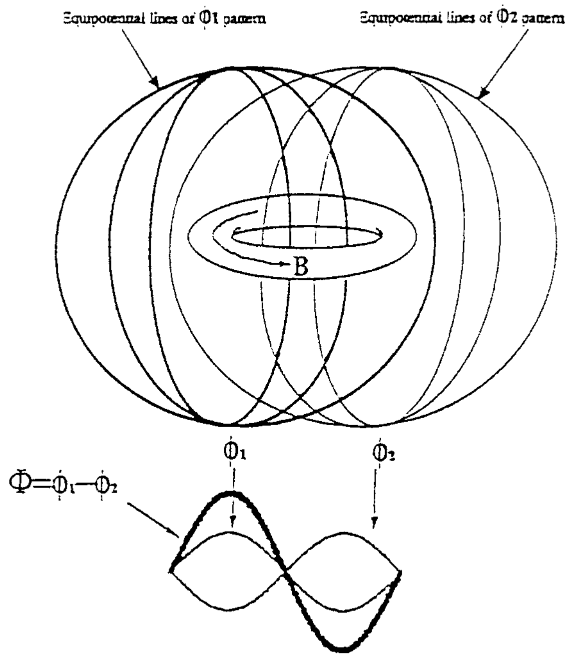

Fig. 5. The toroidal solenoid is viewed as a space for mapping two A vector potential alternating patterns ϕ_1 and ϕ_2 onto a differential phase factor wave Φ. In the top figure ϕ_1 and ϕ_2 are shown an exact wavelength out-of-phase on the torus providing maximum generation of Φ (bottom figure). Such out-of-phase condition only occurs for wavelengths of the driving alternating current which are odd multiples of the circumference length of the torus.

toroidal solenoid as standing waves or is transmitted. The question then arises: given an absolute spatial value of the toroidal ring, what is the optimum driving frequency for Φ standing waves and/or propagating waves? Contour lines for the vector potentials, ϕ, and the phase factor, Φ, are shown in the following Fig. 6. Fig. 6D shows that the Φ phase factor waves are spherical waves – either standing spherical waves, or propagating spherical waves.

We suppose a toroidal solenoid with dimensions 12 in. o.d. and 6 in i.d. The average diameter of the current carrying path is 9 in. and the radius is 4.5 in. or 0.1143 m. As the resonance frequencies of interest are related to complex frequencies defined over a torus, the sinusoidal motion is exponential rather than simple harmonic motion. Therefore, with a conductivity $\sigma = 2.5 \times 10^{-8}$ (Ω^{-1}/m), a torus of these dimensions provides a first resonance response at the frequency:

$$\omega_1 = \mathrm{Exp}\left[\xi \times \frac{(2.5 \times 10^{-8}) \times c}{2\pi \times 0.1143}\right], \tag{10}$$

where $c = 2.997925 \times 10^8$, the wavelengths of the resonance conditions are odd multiples of the circumference of the torus, R, and in the above case, $\lambda_{\max} = 2\pi R/n$, $n = 1$. ξ is a variable the value of which depends on coil winding. This resonance response is not a maximum gain response but a low impedance resonance. If the wavelength is an even

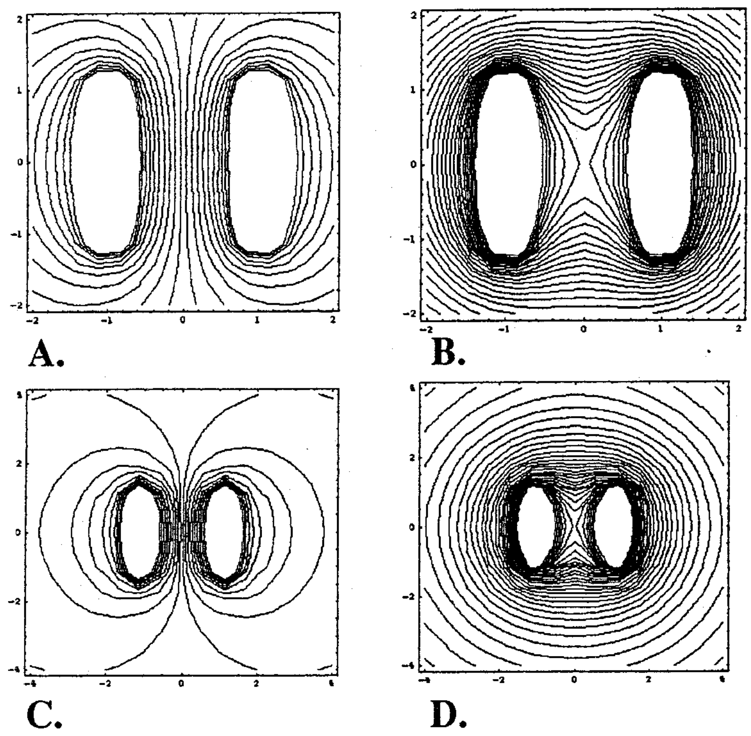

Fig. 6. Contourlines for the ϕ potentials surrounding a torus antenna are shown in (A), and at a greater distance (C). Contourlines for the Φ differential phase factor surrounding a torus antenna are shown in (B). and at a greater distance in (D). (D) indicates that the Φ differential phase factor waves are spherical waves – either standing spherical waves, or propagating spherical waves.

multiple of the circumference of the torus, e.g., $\lambda_{\min} = 2\pi R/m$, $m = 2, 4, 6, \ldots$, there is wave cancellation and a minimum differential phase factor, Φ. Such cancellation provides a minimum gain response, i.e., a maximum impedance response.

In general, therefore, the maximum resonance gain condition (minimum impedance) is:

$$\omega_{\max} = \mathrm{Exp}\left[\xi \times \frac{n \times \sigma \times c}{2\pi \times R}\right], \quad n = 1, 3, 5, \ldots \tag{11}$$

and the minimum resonance gain condition (maximum impedance) is:

$$\omega_{\min} = \mathrm{Exp}\left[\xi \times \frac{m \times \sigma \times c}{2\pi \times R}\right], \quad m = 2, 4, 6, \ldots \tag{12}$$

The resonance maximima and minima for the particular torus of Fig. 2 are shown in Table 1, above. Fig. 7, below, indicates similar information concerning low and high impendance resonances.

Another variable affecting resonance performance is whether the torus is tightly or loosely wound. In the case of the tightly wound torus, the B field is contained on the ring. The field will not be completely contained when the ring is loosely wound. The winding will affect the availability of all possible resonances through the variable ξ. Therefore,

Table 1. Resonance conditions for a toroidal antenna: 12 in. o.d. and 6 in. i.d. (R = 4.5 in. = 0.1143 m) $\xi = 0.2$.

$\omega_{\max}$ and $\omega_{\min}$	$k = l/\lambda = n\ or\ m$ $l = 2\ \pi R = 0.7182$ meters	State
8.06 Hz	$k = l/\lambda = n = 1$ $\lambda = 0.718$ m	Resonance
65.00 Hz	$k = l/\lambda = m = 2$ $\lambda = 0.359$ m	Null
524 Hz	$k = l/\lambda = n = 3$ $\lambda = 0.240$ m	Resonance
4.23 KHz	$k = l/\lambda = m = 4$ $\lambda = 0.180$ m	Null
34.06 KHz	$k = l/\lambda = n = 5$ $\lambda = 0.143$ m	Resonance
274.6 KHz	$k = l/\lambda = m = 6$ $\lambda = 0.120$ m	Null
2.214 MHz	$k = l/\lambda = n = 7$ $\lambda = 0.106$ m	Resonance
17.85 MHz	$k = l/\lambda = m = 8$ $\lambda = 0.090$ m	Null
143.9 MHz	$k = l/\lambda = n = 9$ $\lambda = 0.080$ m	Resonance
1.160 GHz	$k = l/\lambda = m = 10$ $\lambda = 0.072$ m	Null
9.336 GHz	$k = l/\lambda = n = 11$ $\lambda = 0.065$ m	Resonance
75.42 GHz	$k = l/\lambda = m = 12$ $\lambda = 0.060$ m	Null
.....etc.	.....etc.	.....etc.

a tightly wound torus (Fig. 8A) will exhibit more resonances than a loosely wound torus (Fig. 8B).

The alternating Φ standing waves/transmissions from any physically sized toroidal solenoid, for the wire winding addressed, can be optimized by choice of driving frequencies at the odd resonances with other frequencies nonoptimum. These Φ waves correspond to (low energy) gauge fields or phase factors, and are or $SU(2)$ symmetry form.

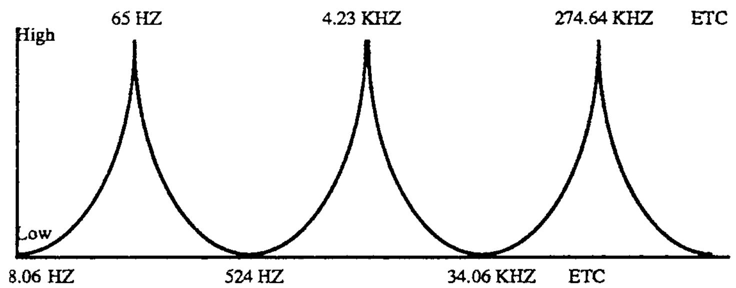

Fig. 7. Representative low and high impedance resonances on the torus of 12 in. o.d. and 6 in. i.d., $x = 0.2$.

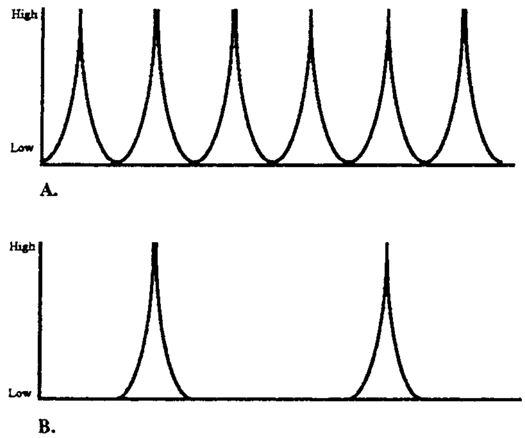

Fig. 8. Representative low and high impedance resonances for tightly wound (A) and loosely wound (B) tori.

The following empirical evidence (Figs. 9 and 10) supports these predictions. Fig. 9 is a plot of range data obtained using a tightly wound torus antenna as a transmit antenna. The plot indicates a $1/r^2$ (one-way) dependence (as opposed to a $1/r$). A less tightly wound torus may exhibit an extended range. There are 15 resonances, 10.1–118 MHz, which are plotted in Fig. 10 as filled circles. These data can be predicted fairly well by:

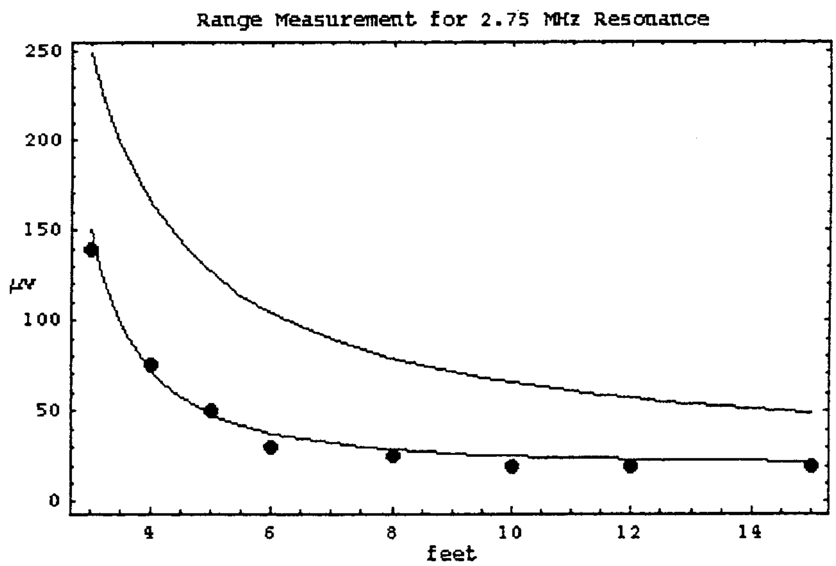

Fig. 9. Upper curve: $400 \times \frac{1}{r - 1.25} + 20$; Lower curve: $400 \times \frac{1}{(r - 1.25)^2} + 20$; Dots: data points. Data courtesy of G. Hathaway and D. Froning.

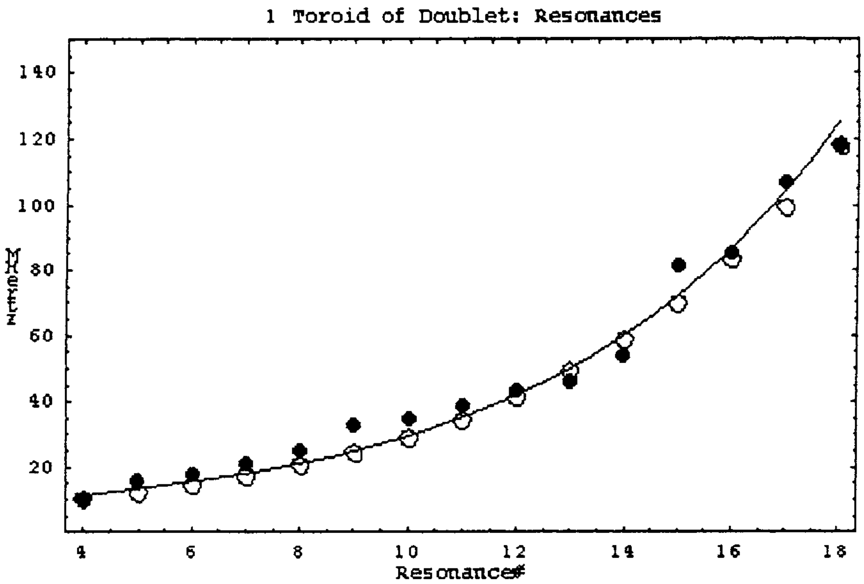

Fig. 10. Some resonances of a singly wound torus. Filled circles: data; unfilled circles: predicted resonances. Data courtesy of G. Hathaway and D. Froning.

$$\text{Resonance} = \text{Exp}\left[\xi \times \frac{n \times \sigma \times c}{2\pi R}\right], \quad n = 1, 3, 5, \ldots \tag{13}$$

where $\sigma = 2.5 \times 10^{-8}\ (\Omega\,\text{m})^{-1}$ is the conductivity; $c = 3 \times 10^{8}$ (m/s) is light speed; $R = 0.1651$ m is the torus radius; and $\xi = 0.0122$. The unfilled circles of Fig. 10 show the predicted resonances.

Caduceous coil (double) winding

In the case of the caduceous coil winding (Fig. 11), the resonances depend, firstly, on the driving connections (usually chosen so that the conductance is opposed in the two windings, or 180° phase difference), secondly, on the number of overlaps in the windings, and, thirdly, on the dimensions of the torus.

Topological mappings

Both the single and the double (caduceous) wound torus permit the mapping of $U(1)$ e.m. fields to $SU(2)$ group symmetry form. Fig. 12 is a representation of this mapping. The Φ phase factor differential is represented as an internal degree of freedom and the mapping itself as a space-time to internal space *fiber connection*. The complete space-time to internal space connection is represented as a *fiber bundle*.

If two sets of twin test particles were introduced to the driven torus – the first of the *first* twin set through the outside of the torus and the second of the *first* twin set through the middle the torus, and the first of the *second* twin set through the middle of the torus and

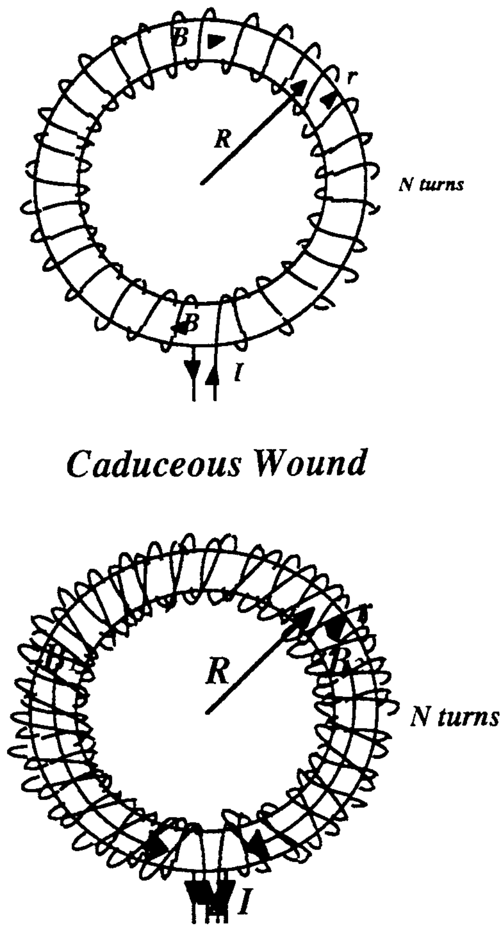

Fig. 11. Single wound torus and caduceous (double wound) torus. In the case of the caduceous winding the driving of the two circuits is 180° out of phase and the winding is left- and right-handed, creating two counterpropagating B fields on the torus.

the second of the *second* twin set through the outside of the torus (Fig. 13) – Aharonov–Bohm effect phases will be detected at overlap locations, $C1$ and $C2$, due to each test particle of each pair being influenced by A fields of opposite polarity. If these phases are compared in a second interferometric level of comparisons, the phase factor field $(C1 - C2)$ will be measured.

Two field mappings are achieved for the resonance condition of the singly wound and the doubly (caduceous) wound torus. A single winding gives a complex number A field representation of the B field (Fig. 14). Either the resonance condition on the single winding or the double (caduceous) winding maps those complex numbers to the sphere S^3.

Details of these mappings are depicted in Fig. 15. Commencing with a half cycle of a sinusoid, $a = b = c = d$, the four half cycles of two monocycles on the torus are formed

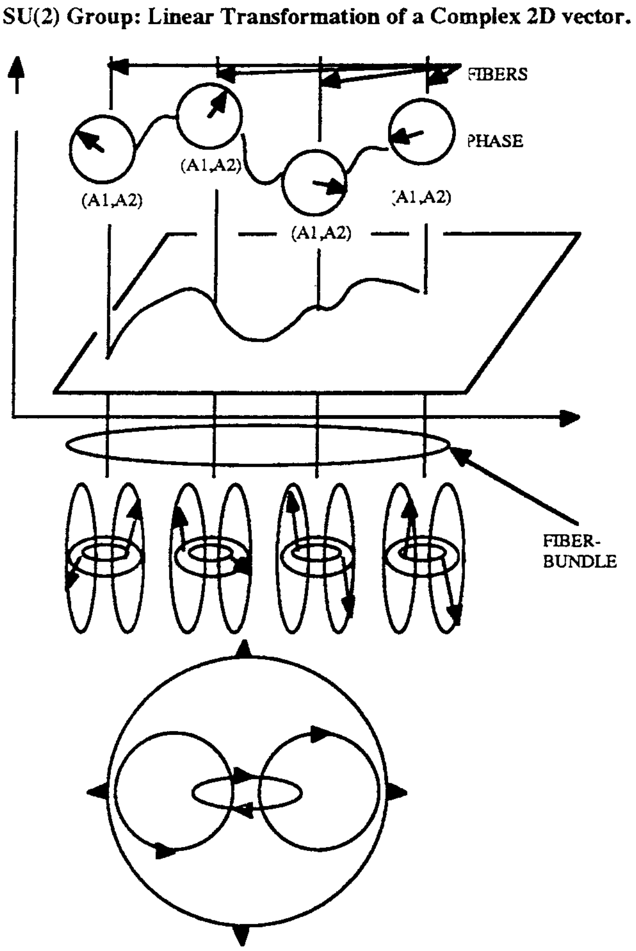

Fig. 12. The topology of a field mapped onto a torus with phase modulation. A fiber connection is shown between the phase in the internal space and the trajectory in space–time. The complete trajectory in internal space is mapped to space–time by a fiber bundle.

(over time), by the I, J, K operations. Thus a complete single quaternion cycle, $H = a + bI + cJ + dK$, is constituted of 2 full sinusoidal monocycles, or 4 half cycles.

The Fig. 15 operation may also be considered a section of a vector bundle over the real numbers of the B field (Fig. 16). At any instant, the driven torus, either singly wound in resonance, or doubly (caduceously) wound, is a section of a vector bundle.

If either the singly wound torus is driven but *not* in resonance, or if the doubly (caduceously) wound torus is driven but *not* 180° out of phase, then the fields on the torus are: (1) closed, (2) exact, (3) of SU (2) symmetry, and (4) *simply* connected. But if the singly wound torus is driven *in resonance*, or if the doubly (caduceously) wound torus is driven 180° out of phase, then the fields on the torus are: (1) closed, but (2) *not* exact, (3) of SU (2)/Z_2 symmetry form (where Z_2 represents the binary integers), and (4) multiply connected (Fig. 17).

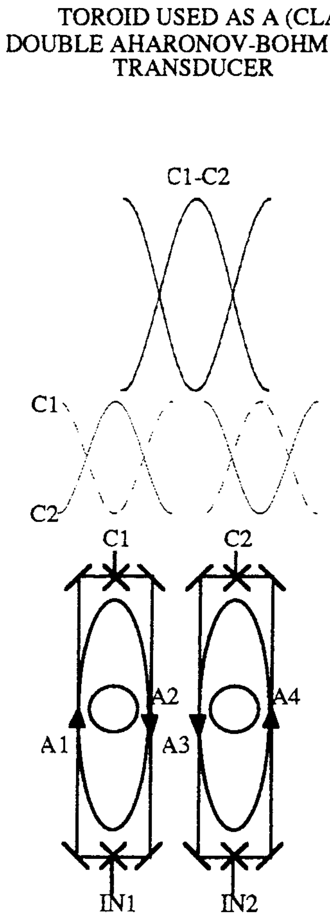

Fig. 13. Opposite fields on a torus used as a double Aharonov–Bohm effect transducer. Two test particles are introduced at *IN*1 and *IN*2 into two interferometers with tracks around opposite sides of a torus antenna. The two arms of the two interferometers interact with the *A* fields *A*1, *A*2, *A*3 and *A*4 which are of alternate directionality. The resulting phase change of the two test particles in the two interferometers are sampled at *C*1 and *C*2. The phase differences $C1 - C2$ is shown at top.

The predictions of (i) omnidirectional radiation patterns (with small indentations at the poles), and (ii) periodic resonances in the driving conditions, have been confirmed by experimental tests (see Figs. 9 and 10, above). Much more extensive testing and confirmation has been carried out by the Center for Industrial Research Applications (CIRA) at West Virginia University, Morgantown, WV, USA.

CIRA has conducted studies and tests of a variety of toroidal antennas (cf. [10, 16]). Fig. 18 shows an example of a multilayer printed circuit board of the Contrawound Toroidal Helical Antenna (CTHA). The CTHA exhibits a gain and impedance spectrum with multiple resonances and which is distinctly different from that of, e.g., a standard dipole antenna.

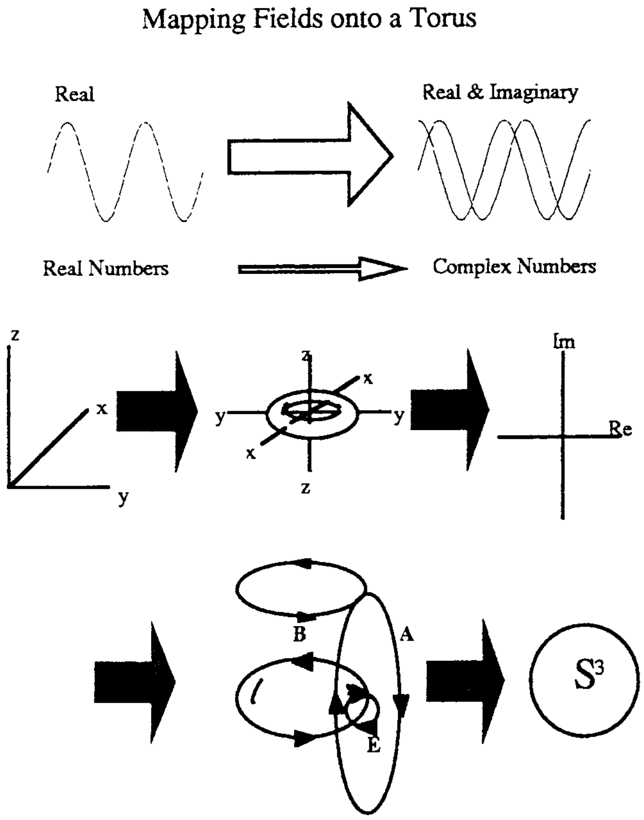

Fig. 14. (Top) Mapping of a real number E field onto a torus antenna results in a complex number representation by the B field due to a first phase representation on a circle. (Bottom) Next, there is a quaternionic representation by the A fields due to two circular representations of the complex number representations by the two A fields on the torus. This results in a phase differential representation. Thus, *in resonance*, the Φ field is the resultant mapping of the E field over S^4.

Figs. 19 and 20 show far field radiations patterns for a typical CTHA. These far field patterns are omnidirectional (with small indentations at the poles) in exact correspondence with the predictions shown in Figs. 3–6 above. The CTHA exhibits other aspects in its radiation pattern which are not accounted for here, but will be addressed in future work. They are:

- Some resonances, e.g., the third, provide a more optimal design than other resonances.
- At various parts of its field pattern, the radiation is present as θ-oriented field energy and at other times as ϕ-oriented field energy (in the spherical polar coordinate system), see Figs. 19 and 20.
- At some resonances, e.g., first and third, the sphere of the far field radiation for a CTHA can produce mixed polarization, i.e., all possibilities of polarization, from left hand circularly polarized, through various degrees of left hand elliptically polarized, to linearly polarized and again through various degrees of right hand elliptical polarization to

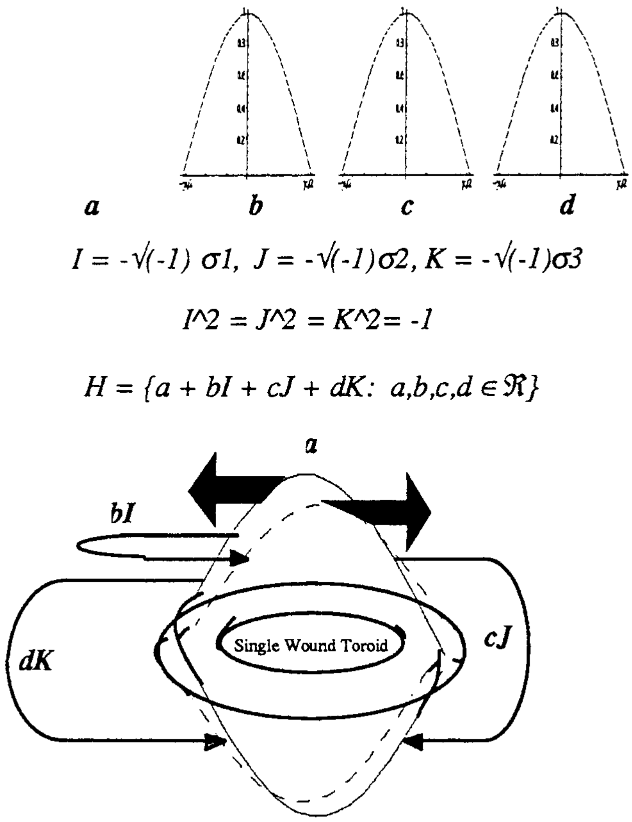

Fig. 15. Representation of a field in quaternionic form on a torus in resonance. The field components are represented as $\pi/2$ sinusoidal base fields: $a = b = c = d$. Mapped onto the torus and with resonance coupling, $a = b = c = d = bI = cJ = dK$. Therefore, a field in resonance on the torus is in quaternionic form: $H = a + bI + CJ + dK$.

right hand circularly polarized. CIRA finds that the typical CTHA regions of circular polarization are 90° either side of the feed, just above and below the horizon. At other resonances, e.g., the second, only linear polarization is obtained.

- Internal interactions of the antenna are the dominant effect on input impedance. Thus the driven antenna is minimally affected by changing environments.

Conclusions

- Treatment of the radiated field of a toroid antenna as two A fields in resonance or a Φ field, and the toroid as a Φ field radiator, results in the prediction of (i) omni-directional radiation patterns (with small indentations at the poles), and (ii) periodic resonances in the driving conditions. These predictions have been confirmed experimentally, giving validity to the fundamental nature of this topological and group theory understanding of the first order determinants of electromagnetic field dynamics.

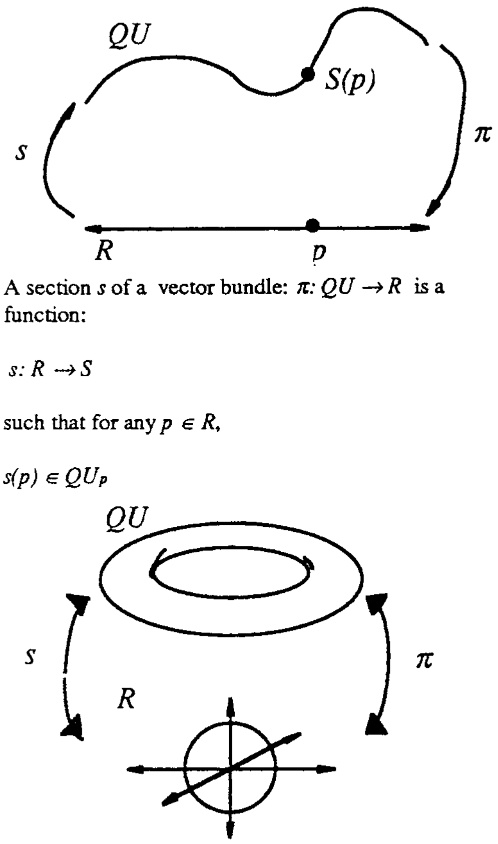

Fig. 16. Section of a vector bundle $\pi : QU \Rightarrow R$. The section assigns to each point in the base space R or the real numbers (i.e., the B field), a quaternion in the fiber over that point. Therefore the quaternionic, QU, space is a bundle over R.

- A field mapped onto a torus by single or caduceous (double) winding maps real numbers of the field into complex numbers at the 2-(differential) form level and quaternions at the 3-(differential) form level.
- $SU(2)$ is the group describing S^3, the unit sphere in the quaternions. However, the radiation of the *in resonance* single winding or double winding mapping on the torus is in quaternionic (or $SU(2)/Z_2$) form, describing S^4, the unit sphere in the quaternions.
- Driven in resonance, e.g., the singly wound toroid driven at wavelengths which are odd multiples of the toroid length and the doubly wound toroid driven at 180° phase lag between the two driving inputs, the radiation is in quaternionic or $SU(2)/Z_2$ form.
- Both (1) the leading and trailing resonance half waves on a singly wound torus and (2) the two mapped fields on a torus in the doubly wound condition are *cohomologous*.
- The field on a torus becomes exact and *simply* connected ($SU(2)$) on S^3 at the 2-form level by mapping two A fields on a torus either singly or doubly wound. Differential forms which are *not exact* are those with integrals equal to zero. Mappings on either the singly wound torus in resonance or the caduceous wound torus *in resonance* results in an

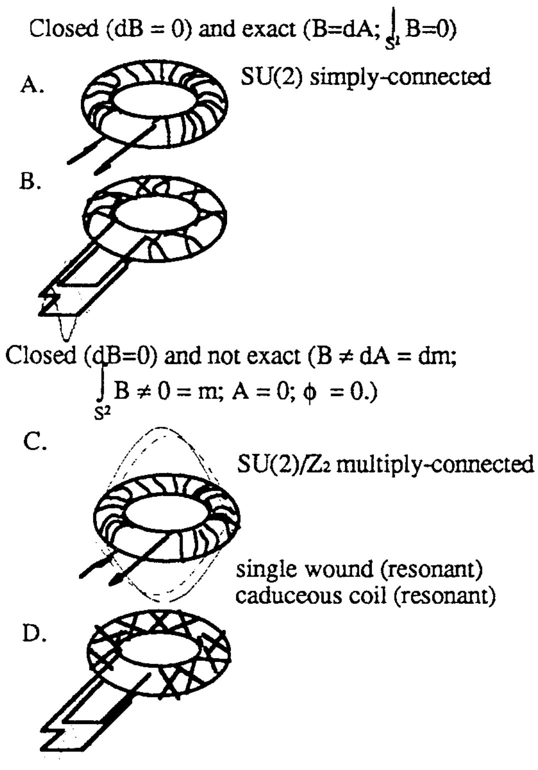

Fig. 17. Fields (*B* fields) on a torus which is either (A) singly or (B) doubly (caduceously) wound, but not in resonance, are: (1) closed (the exterior derivative of the differential form (the *B* field) is zero), (2) exact (the differential form (the *B* field) is the exterior derivative of another differential form (the *A* field), (3) of $SU(2)$ symmetry and (4) *simply* connected. Fields (*B* fields) on a torus which is either (C) singly wound and in resonance or (D) doubly (caduceously) wound and driven 180° out of phase, are (1) closed (the exterior derivative of the differential form (the *B* field) is zero), but (2) not exact (the differential form (the *B* field) is not the exterior derivative of another differential form (the *A* field which is zero), (3) of $SU(2)/Z_2$ symmetry, where Z_2 represents the binary integers and (4) *multiply* connected.

integrated *A* field of zero. Therefore the singly wound *resonance* condition and the doubly wound *resonance* condition result in a *multiply* connected $SU(2)/Z_2$ field on S^4.

- A mapping of a field on a torus which is singly wound in resonance or caduceous wound in resonance is:
  - (A) a homomorphism form $GL(3, C)$ to $SU(2)$, i.e., a representation of C^3 on $SU(2)$.
  - (B) a 3-dimensional complex representation of $SU(2)$ which is a spin-1 representation (quaternionic form).
- A mapping of a field on a torus which is singly wound in resonance or doubly wound in resonance performs an inverse Hodge star operation: $\star\Phi = A_1 \wedge A_2$ and the Φ field is alternatively self-dual and anti-self-dual: $\Phi = \star\Phi$. Self-dual solutions to the Maxwell equations are instantons [13] (Figs. 21 and 23).

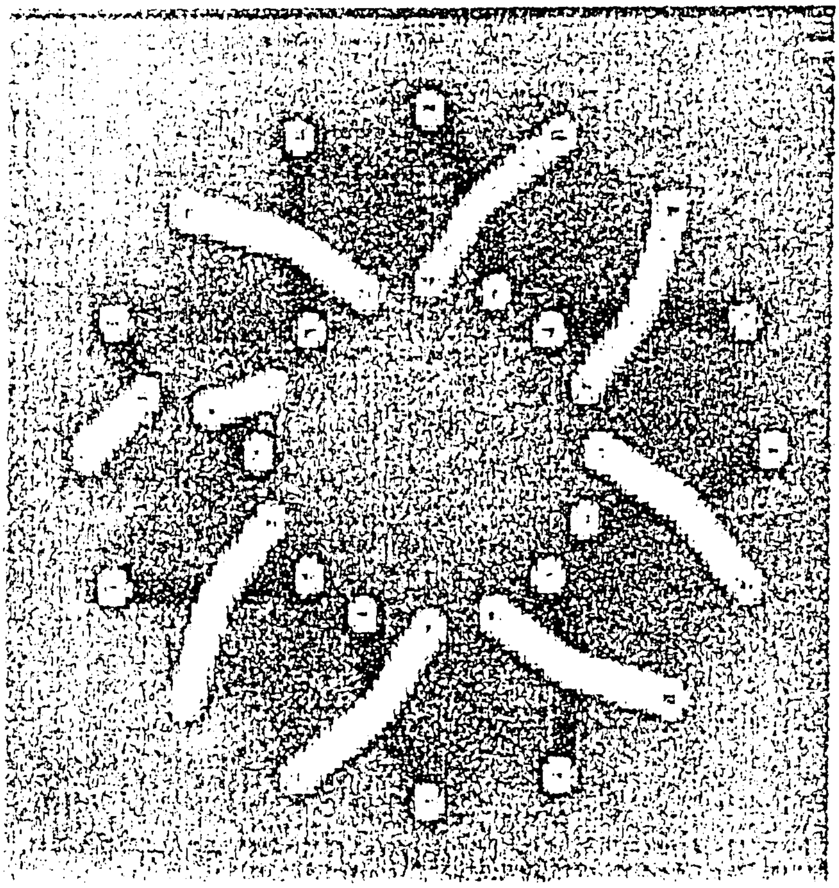

Fig. 18. Multilayer printed circuit board version of the Contrawound Toroidal Helical Antenna (CTHA). From: *Contrawound Toroidal Helical Antenna*, Center for Industrial Research Applications (CIRA), Mechanical and Aerospace Engineering Department, West Virginia University, Morgantown, WV, USA, November 1997, by permission.

- This study has implications far beyond the immediate subject. If the conventional theory of electromagnetism, i.e., 'Maxwell's theory', which is of $U(1)$ symmetry form, is but the simplest local theory of electromagnetism, then those pursuing a unified field theory may wish to consider as a candidate field for unification not only this simple local theory, but other forms of 'conditioned' electromagnetism. As is shown here, other such forms can be either force fields or gauge fields of higher group symmetry, e.g., $SU(2)$ and above.

Appendix

The following definitions are offered in recognition that the concepts of topology and group theory are not well known.