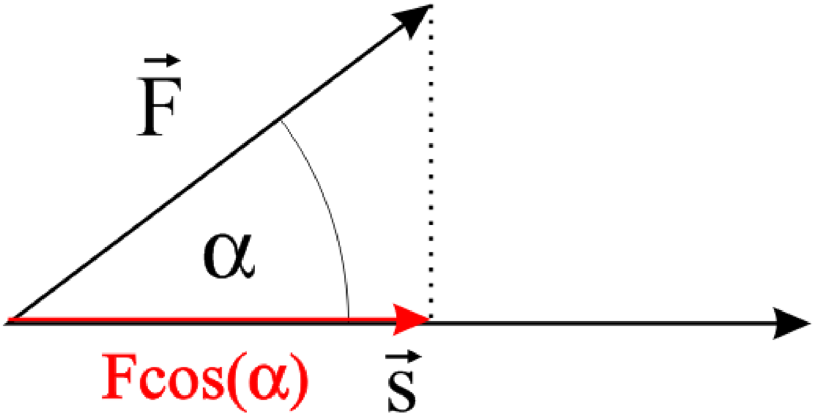
Calculating the work and COP of mechanical devices

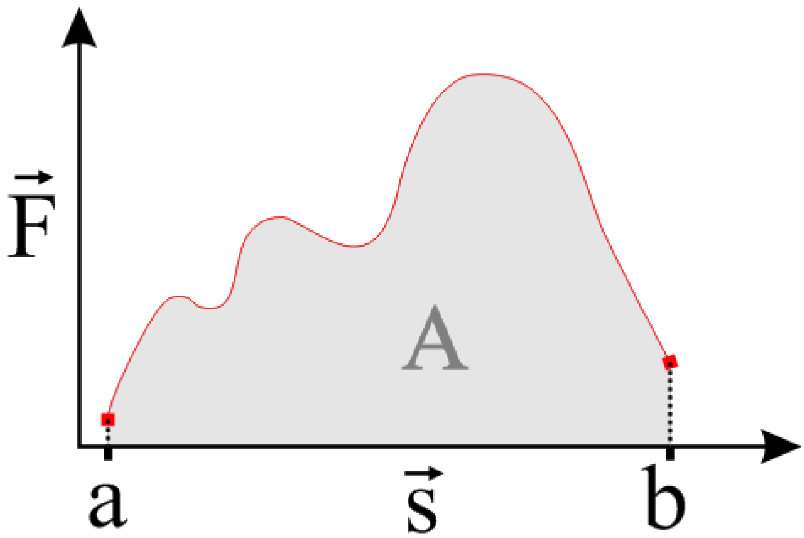
The formula for calculating mechanical work as the scalar product of the force vector $\vec{F}$ and the displacement vector $\vec{s}$ is:

$$W = \vec{F} \cdot \vec{s} = Fs\cos(\alpha)$$

where α is the angle between the direction of the force and the direction of displacement.



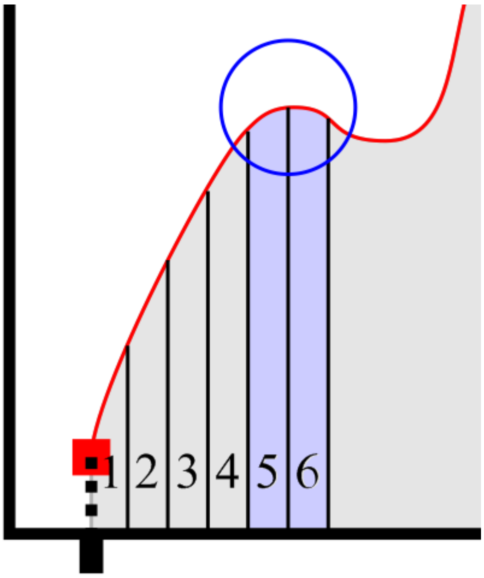
But this equation is valid only if the force remains constant during the complete length of displacement s. In most of the cases during measurements this is not the case, and the measured force is not constant but varies like on this diagram:



If we want to know the total work done by the force $\vec{F}$ during the displacement between points *a* and *b*, then we have to calculate the surface area that is under the red curve, above the *s* axis, and between the two vertical lines at points *a* and *b* (painted grey). If the red curve is a known mathematical function that has an analytic integral, then this surface area *A* which is equivalent with the performed work *W* could be calculated as:

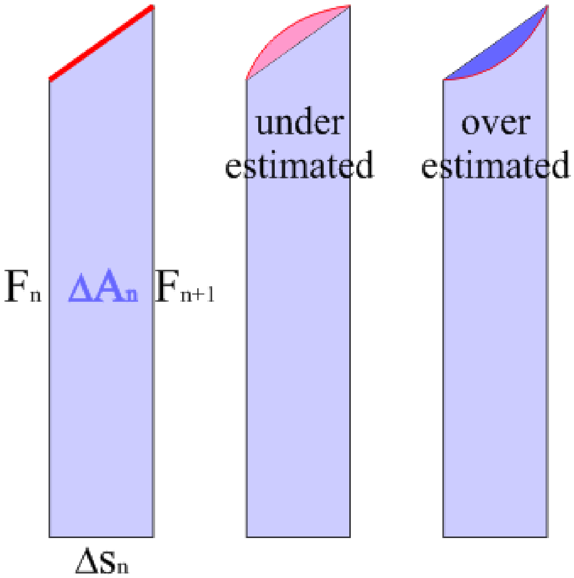
$$W = A = \int_a^b \vec{F} \cdot d\vec{s} = \int_a^b F\cos(\alpha)ds$$

This is rarely the case, so we don't need to worry about calculating analytical integrals. In most of the measurements the curve of the force is not a simple analytical function, therefore we have to calculate the surface area *A* using the method of numerical integration. This is done by breaking up the area A into many very narrow right trapezoids (rectangles with a slanted tops) in order to make the red curve appear on the top of a single trapezoid as a straight line like this:



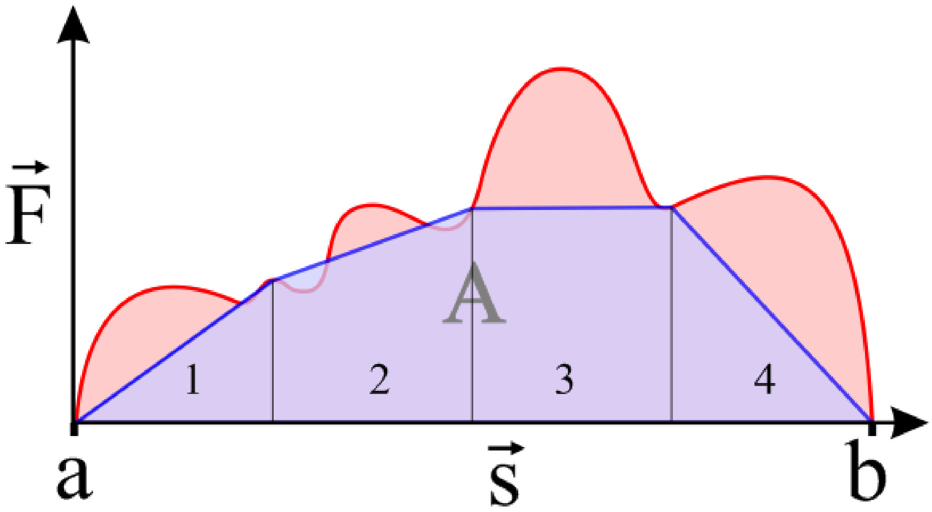
You can see that the red line is almost perfectly straight on top of trapezoids 2 and 3, it can be also considered approximately straight on trapezoids 1 and 4. But it is definitely not straight on top of trapezoids 5 and 6!

Why do we want these curve segments to be straight lines with good approximation? Because we can calculate the surface area of a right trapezoid exactly without any error, and without the need for any approximation like this: if we assume that $\cos\alpha_n = 1$ for all the measurements in this example to simplify the illustration then $\Delta A_n = \Delta s_n (F_n + F_{n+1}) / 2$



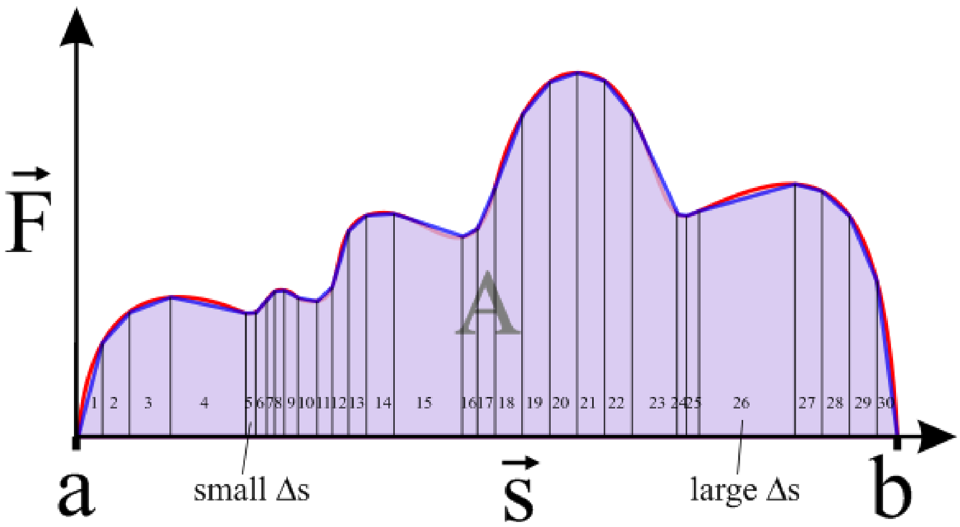
If the force curve on top of a segment is not straight enough, but concave like on the second trapezoid, then our formula will underestimate the real area under the curve (pink area is not getting counted in). If the opposite happens and the force curve is concave, like on top of the second trapezoid, then our formula will overestimate the real area (the blue area which supposed to be left out will be counted in).

The curvature on top of the trapezoids can be reduced to arbitrary small level by simply reducing the width Δs of the segments. There is only one problem with this approach, namely that this way we will have to perform an impractically large number of measurements to cover the total length of displacement. On the other hand if we use large uniform segment width, then this can cause a large error in the final result because the area that we calculate (blue area on the fig. below) will be significantly different from the real area (pink on the image) that supposed to be calculated:



This is the reason why it is not a good practice in general to use uniform Δs segment width. If it is small enough to accurately calculate the sharp curves, then it involves to many measurements. If it is large enough to require a limited practically feasible number of measurements, then its accuracy may be insufficient.

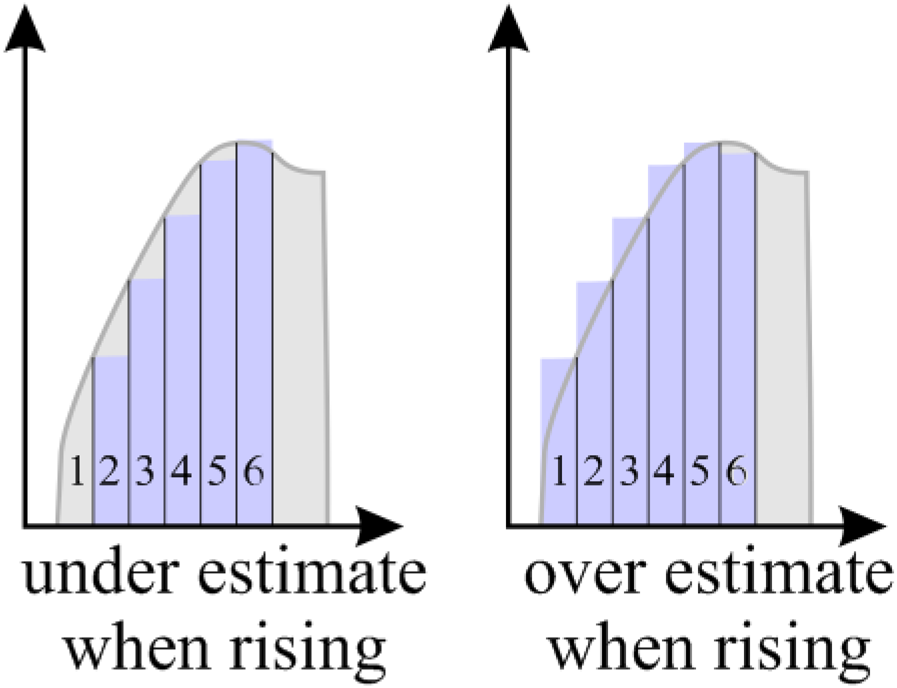
To solution to this problem is to use variable (non-uniform) Δs segment widths. In regions where the curvature of the function is low (or nearly straight) one can use large segment width, and this way significantly reduce the number of measurement points without adversely affecting the accuracy. Where the curvature is significant, the Δs can be reduced to such level that the curve segments can be considered nearly straight (see the image below). This way with a little extra effort the accuracy can be guaranteed.



When such variable Δs is used, the area under the curve can be calculated by a simple summation:

$$A = \sum_{n=1}^{N} (F_n \cos\alpha_n + F_{n+1} \cos\alpha_{n+1}) / 2 \cdot \Delta s_n$$

There are two other alternatives that could be used for the numerical integration of the area under the curve. Both are simplified versions of the above described method, where instead of trapezoids rectangles are used. These versions are slightly less accurate, but simpler to calculate:



The image on the left uses the summation: $A = \sum_{n=1}^{N} F_n \cos\alpha_n \cdot \Delta s_n$

The image on the right uses the summation: $A = \sum_{n=1}^{N} F_{n+1} \cos\alpha_{n+1} \cdot \Delta s_n$

As Δs is decreased, tending towards vanishing small values, all the 3 versions will tend towards giving nearly identical results. However, in practical cases if we use modest a number of measurements, the first method of trapezoids gives the most accurate result.

If during the measurements forces are always measured in tangential direction to the local displacement, then $\cos\alpha = 1$ always, which means that this component can be left out from the above formulas.

The work W performed by the force of the above discussed curve is identical with the calculated surface area A under the curve. In order to calculate the coefficient of performance (COP) of the device, both the input W_{in} and output W_{out} energies (work) must be calculated for a complete cycle of movement. The efficiency can be calculated as:

$$COP = \frac{W_{out}}{W_{in}} \text{; or in \% as: } COP = \frac{W_{out}}{W_{in}} 100\%$$